RIEMANN MAPPING THM

Thm. Let $\Omega \neq \emptyset, \mathbb{C}$ and simply connected; $z_0 \in \Omega$.
Then $\exists!$ conformal map F on Ω:
$$F(\Omega) = \mathbb{D}, \quad F(z_0) = 0, \quad \operatorname{Re} F'(z_0) > 0 \qquad (*)$$

Proof of uniqueness: If F, G satisfy $(*)$ then
$F \circ G^{-1} \in \operatorname{Aut}(\mathbb{D})$ s.t. $F \circ G^{-1}(0) = F(z_0) = 0 \Rightarrow$
is a rotation $F \circ G^{-1}(z) = e^{i\theta} z$, but
$(F \circ G^{-1})'(0) = \frac{F'(z_0)}{G'(z_0)} > 0 \Rightarrow \theta = 2\pi k$ and $F \circ G^{-1} = id$ i.e. $F = G$ ∎

Technical functional/complex analysis lemmata

Def.: $K \subseteq \mathbb{C}$ compact, $\mathcal{F}$ a family of functions from $K \to \mathbb{C}$.
$\mathcal{F}$ is said to be equicontinuous on K if
$$\forall \varepsilon > 0 \ \exists \delta > 0 \mid |f(z) - f(w)| < \varepsilon, \ \forall z, w \in K, |z-w| < \delta, \ \forall f \in \mathcal{F}. \qquad (Eq)$$
Equibounded if $\exists M \mid |f| \leq M \ \forall f \in \mathcal{F}$.

Lemma 1 (Ascoli-Arzelà THM on $\mathbb{C}$) [$\{f_n\}$ equibdd, equicont on K $\Rightarrow \exists \{f_{n_k}\}$ unif. conv.]
Let K be a $(\neq \emptyset)$ compact set in $\mathbb{C}$ and $\{f_n\}$ a sequence of equibounded and equicontinuous functions on K. Then, $\exists f_{n_k}$ uniformly convergent on K.

Proof Let $\{w_j\} \subseteq K$ a sequence dense in K (e.g. $\mathbb{Q}^2 \cap K$).
Since $|f(w_j)| \leq M$ (equibdd) $\overset{B-W}{\Rightarrow} \exists \{f_j^{(1)}\} < \{f_j\} \mid f_j^{(1)}(w_1)$ converges
(subsequence), $\exists \{f_j^{(2)}\} < \{f_j^{(1)}\} \mid \{f_j^{(2)}(w_2)\}$ converg. etc. Notice $\{f_j^{(k)}(z_\ell)\}$ converges $\forall \ell \leq k$.
Let $\{f_{n_j}\} = f_j^{(j)}$ (note $\{f_{n_j}\} < \{f_j^{(k)}\} \ \forall k \leq j$) $\Rightarrow f_{n_j}(w_k)$ converges $\forall k$.
Let $\varepsilon > 0$ and δ s.t. (Eq) holds with $\varepsilon/3$. $\{D_\delta(w_j)\}$ is an open cover of $K \Rightarrow \exists j_1, \dots, j_s$:
$K \subseteq D_\delta(w_{j_1}) \cup \dots D_\delta(w_{j_s})$. Let $N : |f_{n_j}(w_{j_\ell}) - f_{n_k}(w_{j_\ell})| < \varepsilon/3 \ \forall \ 1 \leq \ell \leq s, \forall j, k \geq N$

$\{g_k\}$ is uniformly convergent in $K \iff \forall \varepsilon > 0 \ \exists k_0 \mid k, h \geq k_0 \Rightarrow$
$|g_k(z) - g_h(z)| < \varepsilon \iff g(z) := \lim g_k(z) \ \exists \ \forall z$ and $g_k \to g$ unif. eq.
If g_k are continuous, so is g: $|g(z) - g(z_0)| \leq |g(z) - g_{k_0}(z)| + |g_{k_0}(z) - g_{k_0}(z_0)| + |g_{k_0}(z_0) - g(z_0)|$

Then, $\forall z \in K, \exists \ell \mid z \in D_\delta(w_{j\ell}) \Rightarrow |z - w_{j\ell}| < \delta$ and if $j, k \geq N$.

$$|f_{n_j}(z) - f_{n_k}(z)| \leq |f_{n_j}(z) - f_{n_{j\ell}}(w_{j\ell})| + |f_{n_{j\ell}}(w_{j\ell}) - f_{n_k}(w_{j\ell})|$$
$$+ |f_{n_k}(w_{j\ell}) - f_{n_k}(z)| < \varepsilon . \square$$

② **Lemma 2** (topological lemma) Let $\Omega \neq \emptyset$ open set in $\mathbb{C}$. $\exists \{K_j\}_1^\infty$ s.t. $K_j \subseteq \Omega$, K_j compact ; $K_j \subseteq \mathring{K}_{j+1}$; $\cup K_j = \Omega$.

Proof. Assume Ω is bdd. Then, define $K_j = \{z \in \Omega \mid dist(z, \partial\Omega) \geq \frac{1}{j}\}$

Clearly, $K_j \subseteq \mathring{K}_{j+1}$ (if dist$(z, \partial\Omega) \geq \frac{1}{j} > \frac{1}{j+1}$), K_j is compact.
If $z \in \Omega$, $\exists r$: $D_r(z) \subseteq \Omega \Rightarrow$ dist$(z, \partial\Omega) \geq r$ thus that $\frac{1}{j} < r \Rightarrow z \in K_j$.

If Ω is unbounded, let $\Omega_n = \Omega \cap \{|z| < n\}$, and let $\{K_j^{(n)}\}_j$ be as above $(K_j^{(n)} \subseteq \mathring{K}_{j+1}^{(n)}, \cup_j K_j^{(n)} = \Omega_n)$. Now, let $K_j := \bigcup_{1 \leq n \leq j} K_j^{(n)}$. $\blacksquare$

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③ **Montel's theorem**

Def. Ω open set in $\mathbb{C}$. A family of functions $\mathcal{F}$ is called normal if $\forall \{f_n\} \subseteq \mathcal{F} \ \exists \{f_{n_k}\}$ unif. ly convergent on every $K \subseteq \Omega$ compact.

Lemma 3 (Montel's theorem) Let Ω be an open set and $\mathcal{F}$ a family of holomorphic functions uniformly bdd on compact set (i.e. $\forall K \subseteq \Omega$ compact $\exists M \mid |f(z)| \leq M, \forall z \in K, \forall f \in \mathcal{F}$). Then $\mathcal{F}$ is a normal family.
(i.e. from any sequence $\{f_n\}$, $f_n \in \mathcal{F}$, $\exists f_{n_k}$ unif. ly conv. on compact sets of Ω)

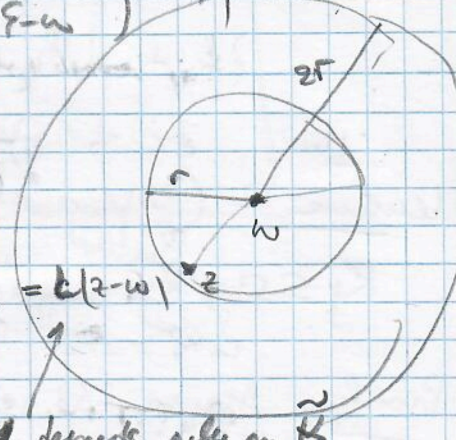
Proof. We first prove that $\mathcal{F}$ is equicont. unif. on compact sets.
Let K be a compact set in Ω. By Lemma 2 $\exists \tilde{K}$ compact in $\Omega \mid K \subseteq \mathring{\tilde{K}}$.
Let $r = \frac{1}{3}$dist$(\partial K, \partial \tilde{K}) > 0$ and let M s.t.
$$|f(z)| \leq M, \ \forall z \in \tilde{K}, \forall f \in \mathcal{F}.$$

Let $z, w \in K$ with $|z-w| < r$ and let $\gamma = \partial D_{2r}(w)^+ \subseteq \mathring{\tilde{K}}$

By Cauchy formula, $\forall f \in \mathcal{F}$,

$$|f(z) - f(w)| = \left|\frac{1}{2\pi i}\int_\gamma f(\xi)\left(\frac{1}{\xi - z} - \frac{1}{\xi - w}\right) d\xi\right|$$

$$= \left|\frac{(z-w)}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi-z)(\xi-w)}\right|$$



$$\le |z-w|\, M \cdot \cancel{2\pi} \cdot \frac{1}{\cancel{r} \cdot 2r} = \frac{M}{r}|z-w| = c|z-w|$$

$|\xi - z| \ge |\xi - w| - |w-z| = 2r - |w-z| > r$

c depends only on $\tilde{K}$

Thus f is unifly Lip on K with Lip. const c $\Rightarrow$ ~~is~~

$\mathcal{F}$ is equicontinuous on compact sets of Ω.

(iii) Let $\{f_n\}$ be a sequence of functions in $\mathcal{F}$.

By A-A (Lemma 1) we know that $\forall$ K compact in Ω $\exists$ $\{f_{n_j}\}$ unifly conv. on K.

Now, let $\{K_j\}$ a sequence of compact sets in Ω as in Lemma 2. Let $\{f_j^{(1)}\} \subset \{f_n\}$ conv. unifly on K_1, $\{f_j^{(2)}\} \subset \{f_j^{(1)}\}$ conv. uniformly on K_2 (and on K_1) etc. then $g_j = \{f_j^{(j)}\}$ is a subsequence of $\{f_n\}$ converging unifly on every K_j.

Now, if K is compact $\overset{\text{by lemma 2}}{\exists}$ j $|$ $K \subseteq K_j$.

and the lemma follows.

Since $\cup K_j = \Omega$ $\forall x \in K$ $\exists j$: $x \in K_j \subseteq \mathring{K}_{j+1}$ - so $j_x := j+1$

By compactness $\exists$ $j_1 \ldots j_s$ | $K \subseteq K_{j_1} \cup \ldots \cup K_{j_s} = K_{j_{max}}$. $j=$

④ Injectivity

Lemma 4 Ω region, $\{f_n\}$ injective holomorphic in Ω s.t. $\{f_n\}$ converges unif'ly on compacts in Ω to f. Then, f is either injective or constant.

Proof By contradiction: assume f is neither injective nor constant. Then $\exists\ z_1 \neq z_2 \mid f(z_1) = f(z_2)$.

Let $g_n(z) := f_n(z) - f_n(z_1)$. Then z_1 is the only zero of g_n (since f_n is injective) and

$g_n \longrightarrow g(z) := f(z) - f(z_1)$ unif-ly on compacts

g is not identically zero ($\Rightarrow f \equiv f(z_1) = $ const)

so z_2 is an isolated zero (if not it would be identically 0 since Ω is connected). By argument principle

$$1 = \frac{1}{2\pi i} \int_\gamma \frac{g'(\xi)}{g(\xi)} \quad \text{for } \gamma = \partial D_r(z_2)^{\uparrow}, \ r \text{ small enough so that } g \neq 0 \text{ on } \overline{D_r(z_2)} - \{z_2\}.$$

$\frac{1}{g_n} \to \frac{1}{g}$ uniformly on γ thus.

$$1 = \frac{1}{2\pi i} \int_\gamma \frac{g'(\xi)}{g(\xi)} = \lim_{n\to\infty} \frac{1}{2\pi i} \int \frac{g_n'(\xi)}{g_n(\xi)} = 0 \quad \text{contradiction.} \ \square$$

g_n has no zeros for r small enough so that $z_1 \notin \overline{D_r(z_2)}$ $(z_1 \neq z_2)$

$$\frac{1}{g} - \frac{1}{g_n} = \frac{g_n - g}{g g_n}$$

and $|g| \geq \underset{\min_\gamma |g|}{\overset{\shortparallel}{\delta}} > 0$ on γ

and for n big enough $|g_n| = |g - (g - g_n)| \geq \delta - \frac{\delta}{2}$. $\square$

$\left(|g - g_n| \leq \frac{\delta}{2}\right)$

(5) <u>Lemma 5</u> Let $\Omega \neq \emptyset$ be simply connected and $g \in H(\Omega)$, $g \neq 0$ on Ω. Then $\exists$ an analytic $f = \log g(z)$ i.e. $\exists\ f \in H(\Omega)$ | $e^f = g$.

<u>Proof.</u> For any $z_0 \in \Omega$ let λ_0 be a logarithm of $g(z_0) \neq 0$ s.t. $e^{\lambda_0} = g(z_0)$ (e.g. $\lambda_0 = \operatorname{Log} g(z_0)$) and define

$$f(z) := \lambda_0 + \int_{\gamma(z_0,z)} \frac{g'}{g}\, d\zeta$$

(by Cauchy thm ↗ depends only on z)

Then, $$(e^{-f} g)' = -e^{-f} \frac{g'}{g} \cdot g + e^{-f} g' = 0$$

on $\Rightarrow$ $$e^{-f} g = \text{const} = e^{-f(z_0)} g(z_0) = e^{-\lambda_0} g(z_0) = 1$$

$\Rightarrow$ $g = e^f$. ■

<u>Rem</u>: If g injective on Ω, then also $\log g$ is:
$\log g(z_1) = \log g(z_2) \Rightarrow g(z_1) = g(z_2) \Rightarrow z_1 = z_2$.

in particular $\log z$ is injective on simply connected domains

Thus a simply connected domain which does not contain the origin cannot have loops which "go around the origin"